INDIAN STATISTICAL INSTITUTE, BANGALORE

B. Math. III Second Semester

Differential Geometry II: Mid-Semester Exam

Duration: 3 hours **Date : March 06, 2015**

Answer any five questions. *Maximum Marks: 100*

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1. *Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. For any $p \in \mathbb{R}^n$ there is a canonical identification $T_p\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ by*
$$\sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}|_p \mapsto (a^1, \dots, a^n).$$
Show that the differential
$$dL|_p : T_p\mathbb{R}^n \to T_{L(p)}\mathbb{R}^m$$
is the map $L : \mathbb{R}^n \to \mathbb{R}^m$ itself, with the identification of the tangent spaces as above.

2. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by*
$$f(x, y) = x^3 + xy + y^3 + 1$$
for which points $p = (0, 0), p = (\frac{1}{3}, \frac{1}{3}), p = (-\frac{1}{3}, \frac{1}{3})$ is $f^{-1}(f(p))$ an embedded sub-manifold in $\mathbb{R}^2$?

3. *The unit sphere S^n in $\mathbb{R}^{n+1}$ is defined by the equation $\sum_{i=1}^{n+1} (x^i)^2 = 1$. For $p = (p^1, ..., p^{n+1}) \in S^n$, show that a necessary and sufficient condition for $v_p = \sum_{i=1}^{n+1} a^i \frac{\partial}{\partial x^i}|_p \in T_p\mathbb{R}^{n+1}$ to be tangent to S^n at p is $\sum_{i=1}^{n+1} a^i p^i = 0$.*

4. *Let $x^1, y^1, ..., x^n, y^n$ be the coordinates in $\mathbb{R}^{2n}$. The unit sphere $S^{2n-1} \subset \mathbb{R}^{2n}$ defined by $\sum_{i=1}^{n} (x^i)^2 + (y^i)^2 = 1$. Show that $X = \sum_{i=1}^{n} -y^i \frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial y^i}$ is nowhere vanishing smooth vector field on S^{2n-1}.*

5. *Find the integral curves of the vector field $X_{(x,y)} = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ on $\mathbb{R}^2$.*

6. *Find all the left invariant vector fields on S^1.*